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Abstract. We construct a measure of $(0,\infty)$ type on the space of C^r mappings, $C^r(M,N)$, and show that it provides a consistent basis for the notion 'generic' and 'exceptional' in the theory of smooth dynamical systems.

1. INTRODUCTION

In order to get a good description of the properties of dynamical systems, we often exclude some set of systems which seem to have singular properties. In such cases, it is important whether we can ignore the excluded set of systems or not. For example, when we consider discrete smooth dynamical systems, we often neglect the systems which have non-hyperbolic periodic points, and the transversality theorem says that such systems are rare. In fact, systems with non-hyperbolic periodic points form a countable union of stratified subsets of codimension one in the space of mappings in some sense. But when we treat more complicated subsets in the space of mappings, we have no idea to judge whether we can neglect them or not. In this paper, we propose a framework to decide negligible subsets of systems, or, in other words, construct a measure of $(0,\infty)$ type on the space of smooth mappings. Of course, we do not claim that our framework is the unique one or the absolute one. There may not be any deductive way to decide such subsets. But we claim that our system is consistent (Theorem B) and that a version of Thom's transversality theorem holds in our framework (Theorem C).

2. Measures on the space of mappings

Let M be a compact C^∞ manifold of dimension m and let

$$\pi : V \to M$$

be a C^∞ vector bundle of dimension p over M. We denote the set of C^r sections of the vector bundle V by $\Gamma^r(V)$, which is endowed with the C^r norm and C^r topology. Then, there are natural inclusions of Banach spaces:

$$\Gamma^0(V) \supset \Gamma^1(V) \supset \Gamma^2(V) \supset \cdots .$$

In this sequence of Banach spaces, each space is dense in the bigger spaces and the Borel σ-algebra on it coincides with the restriction of those on the bigger spaces.

Let $\tau_\varphi : \Gamma^0(V) \to \Gamma^0(V)$ be the translation by $\varphi \in \Gamma^0(V)$. We say a Borel probability measure μ on $\Gamma^0(V)$ is quasi-invariant along the subspace $\Gamma^r(V)$ if $\tau_\varphi(\mu)$ is equivalent to μ for any element $\varphi \in \Gamma^r(V)$, and we denote the set of such measures by $\mathcal{M}_r$. Put $\mathcal{M}_\infty = \cup_{r=0}^{\infty} \mathcal{M}_r$. Remark that the set $\mathcal{M}_r$ is not empty for sufficiently large r. (See the proof of Lemma A.)

Then let us put

$$\tilde{Z}(\Gamma^r(V)) = \{E \in \mathcal{B}(\Gamma^r(V)) \mid \mu(E) = 0 \text{ for any } \mu \in \mathcal{M}_\infty.\},$$

and

$$Z(\Gamma^r(V)) = \bigcap_{\psi \in D(V)} \psi_*(\tilde{Z}(\Gamma^r(V))),$$

where $D(V)$ is the group of C^∞ diffeomorphisms, $\psi : V \to V$, which satisfies $\pi \circ \psi = \pi$ and ψ_* is the action of the element $\psi \in D(V)$ on $\Gamma^r(V)$ such that

$$graph(\psi_*(\phi)) = \psi(graph(\phi)), \qquad \phi \in \Gamma^r(V).$$

Next let us consider the space, $C^r(M,N)$, of C^r mappings from M to a C^∞ manifold N. Choose a C^∞ Riemannian metric on N, and define, for $f \in C^\infty(M,N)$, a homeomorphism

$$\Phi_f : \Gamma^r(f^*TN) \to C^r(M,N)$$

by

$$\Phi_f(h)(x) = \exp_{fx}(h(x))$$

on a neighborhood, U_f, of the zero section. Then the coordinate system

$$\{(\Phi_f, U_f), f \in C^\infty(M,N)\},$$

makes $C^r(M,N)$ a Banach manifold.([**3**])

For the space $C^r(M,N)$, let $Z(C^r(M,N))$ be the family of Borel subsets, $E \subset C^r(M,N)$, such that the set $\Psi_f^{-1}(E \cap \Psi_f(U_f))$ belongs to $Z(\Gamma^r(M,N))$ for every $f \in C^\infty(M,N)$. Since $Z(\Gamma^r(V))$ is invariant under the action of $D(V)$, the definition of $Z(C^r(M,N))$ does not depend on the choice of C^∞ Riemannian metric on N or the choice of U_f's. In this paper, we propose to regard a set of systems $E \subset C^r(M,M)$ as negligible when E belongs to $Z(C^r(M,M))$. At least, we have the following basic facts.

Lemma A. *1) Countable union of elements of the family $Z(C^r(M,N))$ is also contained in $Z(C^r(M,N))$. And if a Borel set E is contained in a set $E' \in Z(C^r(M,N))$, then $E \in Z(C^r(M,N))$.*
2) Any subset $E \in Z(C^r(M,N))$ has no interior with respect to the C^r topology.

From 1) above, we can define a measure m on $C^r(M,N)$ in the following way

$$m(E) = \begin{cases} 0, & \text{if } E \in Z(C^r(M,N)); \\ \infty, & \text{otherwise.} \end{cases}$$

Remark: We can introduce a measure m on the space of vector fields, $\Gamma^r(TM)$, in the same manner *i.e.*

$$m(E) = \begin{cases} 0, & \text{if } E \in Z(\Gamma^r(TM)); \\ \infty & \text{otherwise} . \end{cases}$$

3. Properties of the measure m

As for n-parameter families, we have the following:

Theorem B. *If $m(E) = 0$ for a Borel subset $E \subset C^r(M, N)$, then, for any probability measure λ on $[0,1]^n$, we have*

$$m(S_{E,\lambda}) = 0$$

where

$$S_{E,\lambda} = \{F(x,t) \in C^r(M \times [0,1]^n, N) \mid \lambda\{t \in [0,1]^n \mid F(\cdot,t) \in E\} > 0\}$$

and m is the measure on $C^r(M \times [0,1]^n, N)$ which is constructed as above.

Also the following version of Thom's transversality theorem [**1**] holds.

Theorem C. *Let X be a C^1submanifold of the jet bundle $J^r(M,N)$, then we have*

$$m\{f \in C^{r+1}(M,N) \mid j^r f \text{ is not transversal to } X\} = 0.$$

Remark: See [**1**] for the definition of jet bundles.

The following fact shows that the measure m is copmpatible with the Lebesgue measure (the class of measures which is equivalent to the smooth Riemannian volume). We consider a map, for $q \leq r$,

$$\alpha : M \times C^r(M,N) \longrightarrow J^q(M,N)$$

defined by

$$\alpha(x, f) = j^q f(x).$$

Theorem D. *Let X be a Borel subset of $J^q(M, N)$ with Lebesgue measure zero. Then*

$$m\{f \in C^r(M, N) \mid (j^q f)^{-1}(X) \text{ has positive Lebesgue measure.}\} = 0.$$

4. Proof of Theorems

In the proof below, we always assume $N = \mathbf{R}^p$, and, thus, $C^r(M, N) = \Gamma^r(M \times \mathbf{R}^p)$. It is a routine to extend our proof to the case $N \neq \mathbf{R}^p$.

Proof of lemma A: The claim 1) is self-evident. In order to prove 2), let us introduce Sobolev spaces:

$$W^r(M, \mathbf{R}^p) = \{f \in \Gamma^0(M, \mathbf{R}^p) \mid d^r f \in L^2\}.$$

If s is sufficiently larger than r, then the inclusion map

$$W^s(M, \mathbf{R}^p) \subset W^r(M, \mathbf{R}^p)$$

is a Hilbert-Schmidt operator. Therefore, we can construct a Gaussian measure on the space $W^r(M, \mathbf{R}^p)$ which is quasi-invariant along the space $W^s(M, \mathbf{R}^p)$ and takes positive value for every open set on $W^r(M, \mathbf{R}^p)$. (See [**2**] or the proof of Lemma E in the last section.) Since we have the following continuous inclusions, by Sobolev's embedding theorem,

$$\Gamma^{r-[m/2]-1}(M, \mathbf{R}^p) \supset W^r(M, \mathbf{R}^p) \supset W^s(M, \mathbf{R}^p) \supset \Gamma^s(M, \mathbf{R}^p),$$

we can get the claim 2).

Proof of theorem B: Let us define maps

$$\xi : C^0(M \times [0,1]^n, \mathbf{R}^p) \times [0,1]^n \longrightarrow C^0(M, \mathbf{R}^p)$$

and

$$\xi_t : C^0(M \times [0,1]^n, \mathbf{R}^p) \longrightarrow C^0(M, \mathbf{R}^p)$$

by

$$\xi(F(\cdot,\cdot),t) = F(\cdot,t)$$

and

$$\xi_t(F(\cdot,\cdot)) = F(\cdot,t).$$

For any Borel probability measure μ on $C^0(M \times [0,1]^n, \mathbf{R}^p)$ which is quasi-invariant along $C^r(M \times [0,1]^n, \mathbf{R}^p)$, the measure $\xi_t(\mu)$ on $C^0(M, \mathbf{R}^p)$ is quasi-invariant along $C^r(M, \mathbf{R}^p)$. Because, for any $\varphi \in C^r(M, \mathbf{R}^p)$, the following diaglam commutes:

$$\begin{array}{ccc} C^0(M \times [0,1]^n, \mathbf{R}^p) & \xrightarrow{\xi_t} & C^0(M, \mathbf{R}^p) \\ \Big\downarrow{\scriptstyle \tau_{\tilde{\varphi}}} & & \Big\downarrow{\scriptstyle \tau_{\varphi}} \\ C^0(M \times [0,1]^n, \mathbf{R}^p) & \xrightarrow{\xi_t} & C^0(M, \mathbf{R}^p) \end{array}$$

where $\tilde{\varphi} = \varphi \circ \pi' \in C^r(M \times [0,1]^n, \mathbf{R}^p)$. ($\pi' : M \times [0,1]^n \to M$ is the projection.) Thus we have,

$$\mu(\xi_t^{-1}(E)) = (\xi_t\mu)(E) = 0$$

Let ψ be an element of $D((M \times [0,1]^n) \times \mathbf{R}^p)$ and put $\tilde{\psi} = \pi'' \circ \psi \circ \iota_t \in D(M \times \mathbf{R}^p)$ where $\pi'' : M \times [0,1]^n \times \mathbf{R}^p \to M \times \mathbf{R}^p$ is the projection and $\iota_t : M \times \mathbf{R}^p \to M \times [0,1]^n \times \mathbf{R}^p$ is the map defined by $\iota_t(x,v) = (x,t,v)$. (Here we consider $M \times [0,1]^n \times \mathbf{R}^p$ and $M \times \mathbf{R}^p$ as trivial vector bundles with $\mathbf{R}^p$ their fiber.) Then the following diagram commutes:

$$\begin{array}{ccc} C^0(M \times [0,1]^n, \mathbf{R}^p) & \xrightarrow{\xi_t} & C^0(M, \mathbf{R}^p) \\ \Big\downarrow{\scriptstyle \psi_*} & & \Big\downarrow{\scriptstyle \tilde{\psi}_*} \\ C^0(M \times [0,1]^n, \mathbf{R}^p) & \xrightarrow{\xi_t} & C^0(M, \mathbf{R}^p) \end{array}$$

and, from this, we have

$$\psi_*(\mu)(\xi_t^{-1}(E)) = \xi_t(\mu)(\tilde{\psi}_*^{-1}(E)) = 0.$$

Therefore, for any Borel probability measure λ on $[0,1]$, we have

$$\psi_*(\mu) \times \lambda(\xi^{-1}(E)) = 0$$

and then, by Fubini's theorem,

$$\psi_*(\mu)(S_{E,\lambda}) = 0.$$

The last expression implies the theorem.

Proof of theorem C: Take a chart on an open set $V \subset M$, $\varphi : V \to \mathbf{R}^m$, and let U be an open set whose closure is contained in V. Let $\rho : \mathbf{R}^m \to [0,1]$ be a C^∞ function on $\mathbf{R}^m$ such that

$$\rho(x) = \begin{cases} 1, & \text{on a neighborhood of the closure of } \varphi(U) \text{ ;} \\ 0, & \text{off } \varphi(V). \end{cases}$$

We denote, by B, the space of polynomial mappings of $\mathbf{R}^m \to \mathbf{R}^p$ of degree r , and define a map

$$\Phi : B \times C^{r+1}(M, \mathbf{R}^p) \to C^{r+1}(M, \mathbf{R}^p)$$

by

$$\Phi(b,f)(x) = \begin{cases} f(x) + \rho(\varphi(x))b(\varphi(x)), & \text{if } x \in V; \\ f(x), & \text{otherwise.} \end{cases}$$

For any $f \in C^{r+1}(M, \mathbf{R}^p)$, the map

$$\Psi_f : B \times U \to J^r(U, \mathbf{R}^p) \subset J^r(M, \mathbf{R}^p)$$

defined by

$$\Psi_f(b,x) = j^r(\Phi(b,f))(x)$$

is a submersion. Therefore, the set

$$X_f = \{(b,x) \in B \times U \mid \Psi_f(b,x) \in X\}$$

is a C^1submanifold in $B \times U$. Remark that the map $j^r(\Phi(b,f))$ is transversal to X on U if and only if the point b is a regular value for the map

$$p : X_f \to B,$$

which is the restriction of the projection $B \times U \to B$ to X_f. From Sard's theorem, we have

$$\lambda\{\, b \in B \mid j^r(\Phi(b,f)) \text{ is not transversal to } X \text{ on } U.\} = 0$$

for any $f \in C^{r+1}(M, \mathbf{R}^p)$, where λ is a probability measure on B which is equivalent to the smooth Riemannian volume. Therefore,

$$\begin{aligned}
&\Phi(\lambda \times \mu)\{\, f \in C^{r+1}(M, \mathbf{R}^p) \mid j^r f \text{ is not transversal to } X \text{ on } U.\} \\
&\quad = \lambda \times \mu\{\, (b,f) \in B \times C^{r+1}(M, \mathbf{R}^p) \mid j^r(\Phi(b,f)) \text{ is not transversal to } X \text{ on } U.\} \\
&\quad = 0
\end{aligned}$$

for any Borel probability measure μ on $C^{r+1}(M, \mathbf{R}^p)$. On the other hand, in case $\mu \in \mathcal{M}_\infty$, $\Phi(\lambda \times \mu)$ is equivalent to μ , because

$$\Phi(\lambda \times \mu)(E) = \int_B \mu(\tau_{-(\rho \cdot b)\circ\varphi}(E))\, d\lambda(b)$$

for any Borel set E in $C^{r+1}(M, \mathbf{R}^p)$. Therefore, we have proved that the set

$$T_{X,U} = \{ f \in C^{r+1}(M, \mathbf{R}^p) \mid\ j^r f \text{ is not transversal to } X \text{ on } U\}$$

belongs to $\tilde{Z}(C^{r+1}(M, \mathbf{R}^p))$. Since our argument above do not change under the action of $D(M \times \mathbf{R}^p)$, the set $T_{X,U}$ belongs to $Z(C^{r+1}(M, \mathbf{R}^p))$. From this and lemma A 1), we can see the theorem.

Proof of theorem D: Let $U, V, \varphi, \rho, B, \Phi$ be those in the proof of theorem C above and let λ be a probability measure on M which is equivalent to the smooth Riemannian volume. For sufficiently small $y \in \mathbf{R}^m$, we can define a diffeomorphism $t_y : M \to M$ by

$$t_y(x) = \begin{cases} \varphi^{-1}(\, \rho(\varphi(x)) \cdot y + \varphi(x)\,), & \text{if } x \in V; \\ x & \text{otherwise.} \end{cases}$$

For $v = (y, b) \in \mathbf{R}^m \times B$ with y sufficiently small, let us define a mapping

$$\gamma_v : M \times C^r(M, \mathbf{R}^p) \to M \times C^r(M, \mathbf{R}^p)$$

by

$$\gamma_v(x, f) = (t_y^{-1}(x), \Phi(b, f)).$$

Then, there exists a C^∞diffeomorphism

$$\gamma_v' : J^q(M, \mathbf{R}^p) \to J^q(M, \mathbf{R}^p)$$

such that the following diagram commutes:

$$\begin{array}{ccc} M \times C^r(M, \mathbf{R}^p) & \xrightarrow{\alpha} & J^q(M, \mathbf{R}^p) \\ \downarrow{\scriptstyle \gamma_v} & & \downarrow{\scriptstyle \gamma_v'} \\ M \times C^r(M, \mathbf{R}^p) & \xrightarrow{\alpha} & J^q(M, \mathbf{R}^p) \end{array}$$

From this, we can see that

$$\gamma_v'(\alpha(\lambda \times \mu)) \sim \alpha(\lambda \times \mu).$$

for any $v = (y, b) \in \mathbf{R}^m \times B$ with y sufficiently small and $\mu \in \mathcal{M}_\infty$. Since the map γ_v' in the local coordinate on $J^q(U, \mathbf{R}^p)$ is nothing but the translation by the vector v, the above equivalence implies that $\alpha(\lambda \times \mu)$ is equivalent to the smooth Riemannian volume on $J^q(U, \mathbf{R}^p)$. For each $\psi \in D^\infty(M \times \mathbf{R}^p)$, there exists a C^∞diffeomorphism

$$J_\psi^q : J^q(M, \mathbf{R}^p) \to J^q(M, \mathbf{R}^p)$$

which makes the following diagram commutes:

$$\begin{array}{ccc} M \times C^r(M, \mathbf{R}^p) & \xrightarrow{\alpha} & J^q(M, \mathbf{R}^p) \\ \downarrow{\scriptstyle id \times \psi_*} & & \downarrow{\scriptstyle J_\psi^q} \\ M \times C^r(M, \mathbf{R}^p) & \xrightarrow{\alpha} & J^q(M, \mathbf{R}^p) \end{array}$$

Thus we have

$$\lambda \times (\psi_* \mu)(\alpha^{-1}(X)) = \alpha(\lambda \times \mu)\big((J_\psi^q)^{-1}(X)\big) = 0.$$

and, by Fubini's theorem,

$$\psi_* \mu\{f \in C^r(M, \mathbf{R}^p) \mid (j^q f)^{-1}(X) \text{ has positive Lebesgue measure.}\} = 0$$

for any $\psi \in D^\infty(M \times \mathbf{R}^p)$ and any $\mu \in \mathcal{M}_\infty$. This implies the theorem.

5. A Remark

For $\varphi \in C^\infty(M, \mathbf{R}^p)$, let us consider one parameter families of the form

$$f + t \cdot \varphi \qquad t \in \mathbf{R},\ f \in C^r(M, \mathbf{R}^p).$$

Then such set of one parameter families can be considered as a (measurable) partition of the space $C^r(M, \mathbf{R}^p)$ into one dimensional subspaces. The important is the fact that, for $\mu \in \mathcal{M}_\infty$, the conditional measures on each one dimensional subspaces are equivalent to the Lebesgue measure because they are quasi-invariant under the translation. This fact implies that we can get estimates of the value $\mu(E)$ for some $E \subset C^r(M, \mathbf{R}^p)$ from the Lebesgue measure of the set of parameter values, $\{t \in \mathbf{R} \mid f + t\varphi \in E\}$. This is one of the good points of our framework. The following lemma will be useful in proving $m(E) = 0$ for some subset $E \subset C^r(M, N)$. We denote, by $\mathcal{M}'_r$, the set of Borel probability measure $\mu \in \mathcal{M}_r$ satisfying the following condition $(*)$:

$(*)$ For any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \frac{d\tau_\varphi \mu}{d\mu} - 1 \right| < \epsilon, \qquad \mu - a.e.$$

for any $\varphi \in \Gamma^r(V)$ with $\| \varphi \|_{C^r} < \delta$.

Lemma E. *For any measure $\mu \in \mathcal{M}_r$, we can find a measure $\mu' \in \mathcal{M}'_{r+[\frac{5}{2}m]+3}$ which is equivalent to μ .*

Proof: For $s = r + [m/2] + 1$, let us consider the Sobolev space $W^s(V) \subset C^r(V)$.

If there exists a probability measure ν such that

$$(1) \qquad \nu \in \mathcal{M}'_{r+[\frac{5}{2}m]+3}$$

and

$$(2) \qquad \nu(W^s(V)) = 1,$$

then the convolution $\mu' = \mu * \nu$ is also an element of $\mathcal{M}'_{r+[\frac{5}{2}m]+3}$ and equivalent to the measure μ. Therefore let us show the existence of such a measure. First let us consider the case

$$M = T^m = (\mathbf{R}/Z)^m \ (m\text{-torus}), \quad V = T^m \times \mathbf{R}.$$

In this case, we can identify $W^s(V)$ with the Sobolev space of functions, $W^s(T^m) = \{f \in C^0(T^m, \mathbf{R}) \mid d^s f \in L^2(T^m)\}$, with the inner product

$$\langle f, g \rangle_{W^s(T^m)} = \sum_{|u| \leq s} \int_{T^m} d^u f \cdot d^u g \, dx_1 dx_2 \cdots dx_m.$$

Then we can take the following orthonormal basis of the space $W^s(T^m)$:

$$\mathrm{e}(n_1, n_2, \cdots, n_m) = \frac{\mathrm{e}'(n_1, n_2, \cdots, n_m)}{\| \mathrm{e}'(n_1, n_2, \cdots n_m) \|_{W^s(T^m)}}, \qquad n_j \in Z$$

where

$$\mathrm{e}'(n_1, n_2, \cdots, n_m)(x_1, x_2, \cdots, x_m) = \mathrm{e}'(n_1, x_1)\mathrm{e}'(n_2, x_2) \cdots \mathrm{e}'(n_m, x_m)$$

and

$$\mathrm{e}'(n, x) = \begin{cases} 1, & \text{if } n = 0; \\ \sin(2\pi n x), & \text{if } n > 0; \\ \cos(2\pi n x), & \text{if } n < 0. \end{cases}$$

Consider the product space

$$\mathbf{R}^\infty = \prod_{(n_1, \cdots, n_m)} \langle \mathrm{e}'(n_1, \cdots, n_m) \rangle_{\mathbf{R}}.$$

Then we can identify $W^s(T^m)$ with the following subspace of $\mathbf{R}^\infty$:

$$\{\sum_{(n_1,\cdots,n_m)} x(n_1,\cdots,n_m)\mathrm{e}(n_1,\cdots,n_m)\in\mathbf{R}^\infty \mid \sum_{(n_1,\cdots,n_m)} x(n_1,\cdots,n_m)^2<+\infty\}$$

Let $\mu(n_1,\cdots,n_m)$ be a probability measure on the one dimentional subspace, $\langle \mathrm{e}(n_1,n_2,\cdots,n_m)\rangle_{\mathbf{R}}$, of the form

$$(\frac{a}{2})\exp(-a\cdot\mid x\mid)\,dx,$$

where

$$a=(\max_j n_j)^{m+1},$$

and consider the product of them,

$$\nu_1=\prod_{(n_1,\cdots,n_m)\in Z^m}\mu(n_1,\cdots,n_m),$$

on $\mathbf{R}^\infty$. Put, for $c>0$,

$$B_c=\{\sum_{(n_1,\cdot\cdot,n_m)} x(n_1,\cdot\cdot,n_m)\mathrm{e}(n_1,\cdot\cdot,n_m)\mid x(n_1,\cdot\cdot,n_m)<c\cdot(\max_j n_j)^{-m}\}.$$

Then it is easy to see that

$$W^s(T^m)\supset B_c$$

for any $c>0$ and that

$$\begin{aligned}\nu_1(B_c)&=\prod_{(n_1,\cdots,n_m)}\{1-\exp(-c\cdot\max_j n_j)\}\\&\to 1\qquad\text{as }\ c\to+\infty.\end{aligned}$$

Therefore we have

$$\nu_1(W^s(T^m))=1.$$

If $f=\sum x(n_1,\cdots,n_m)\mathrm{e}(n_1,\cdots,n_m)$ is contained in $W^{s+2m+2}(T^m)$, then

$$\sum_{(n_1,\cdots,n_m)}\{(\max_j n_j)^{2m+2}x(n_1,\cdots,n_m)\}^2<c$$

for some constant c , because $d^{2m+2}f \in W^s(T^m)$. And we have, for such f,

$$essup\left(\frac{d\tau_f\nu_1}{d\nu_1}\right) \leq \prod_{(n_1,\cdots,n_m)} essup\left(\frac{d\tau_{x(n_1,\cdots,n_m)e(n_1,\cdots,n_m)}\mu(n_1,\cdots,n_m)}{d\mu(n_1,\cdots,n_m)}\right)$$
$$= \exp\{\sqrt{c}\cdot \sum_{(n_1,\cdots,n_m)} (\max_i n_i)^{-m-1}\}$$
$$\to 1 \qquad \text{as } \| f \|_{W^{s+2m+2}(T^m)} \to 0 \quad (c \to 0)$$

(For the calculation of Radon-Nikodim derivative, see [2], Chapter 3.)
Since $C^{r+[\frac{5}{2}m]+3}(T^m) \subset W^{s+2m+2}$, we have $\nu_1 \in \mathcal{M}'_{r+[\frac{5}{2}m]+3}$. Therefore $\nu = \nu_1$ satisfies the conditions (1) and (2).

In the case

$$M = T^m, V = T^m \times \mathbf{R}^p,$$

we have

$$W^s(M \times \mathbf{R}^p) = W^s(M) \times \overset{\text{p times}}{\cdots} \times W^s(M).$$

Therefore, $\nu_p = \nu_1 \times \overset{\text{p times}}{\cdots} \times \nu_1$ satisfies the conditions (1) and (2).

Finally, let us consider the general case. Take a open covering $\{U_j,\ j = 1, 2, \cdots, d\}$ so that there exist C^∞vector bundle isomorphisms

$$\psi_j : \pi^{-1}(U_j) \to V_j \times \mathbf{R}^p$$

where V_j is an open set on T^m. And, using a partition of unity $\{\phi_j \in C^\infty(M)\}_{j=1}^d$ subordinate to the covering $\{U_j\}$, define the following embedding

$$\Psi : W^s(V) \to \bigoplus_{j=1}^d W^s(T^m \times \mathbf{R}^p)$$
$$f \quad \to \bigoplus_{j=1}^d \psi_j(\phi_j \cdot f)$$

Then the measure $\nu = \Psi^{-1}(\ p\ (\prod_{j=1}^d \nu_p))$ satisfies the condition (1) and (2), where

$$p : \bigoplus_{j=1}^d W^s(T^m \times \mathbf{R}^p) \to \Psi(W^s(V))$$

is the orthogonal projection.

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